Small-scale chaos at low Reynolds numbers

1991 J. Phys. A: Math. Gen. 24 5763

(http://iopscience.iop.org/0305-4470/24/24/012)

J. Phys. A: Math. Gen. **24** (1991) 5763–5773. Printed in the UK

Small-scale chaos at low Reynolds numbers

G A Kuz'min† and A Z Patashinskii‡

† Institute of Thermophysics, 630090 Novosibirsk, USSR
‡ Institute of Nuclear Physics, 630090 Novosibirsk, USSR

Received 2 April 1991

Abstract. A system of dissipative modes in an incompressible flow of space dimensions $d \geqslant 2$ is considered. The self-induced phase chaos is shown to arise in the motions of very small scales, which are fed by the large-scale ones through repeated nonlinear interactions. This property is used to derive the equations for the Fourier amplitudes. Solutions similar to those derived previously for turbulent fluctuations in the dissipation range are obtained. Properties of the short-scale intermittency are analysed. We show that no coherence and intermittency can be built up at asymptotically high wavenumbers.

1. Introduction

The energy of the fully developed turbulence is excited at some scale L, and is transferred through the inertial range to vortices of the Kolmogorov scale η, where it is finally dissipated. Some portion of the turbulent energy penetrates into the dissipation range $k \gg \eta^{-1}$ and produces a rapidly decreasing tail to the turbulence spectrum. In the dissipation range, the actual form of the turbulence spectrum is determined by vortices at low Reynolds number. Nevertheless, the pertinent nonlinearity of the small-scale motions cannot be discarded.

The linear stability analysis gives the following asymptote of the spectrum in the dissipation range (Townsend 1951, Novikov 1962):

$$F(k) \propto \exp[-(\eta k)^2]. \tag{1.1}$$

In statistical theory it has been shown (Kraichnan 1959, Kuz'min 1971, Kuz'min and Patashinskii 1979, Dubovikov and Tatarskii 1986), that the energy transfer via a nonlinear cascade gives the more slowly decaying spectrum

$$F(k) \propto \exp(-\eta k). \tag{1.2}$$

An attempt to solve the problem leads to difficulties that are well known in theories with strong interactions. The main one is the failure of perturbation theory. On the other hand, most of the additional difficulties, which are known to be peculiar to inertial range turbulence theories, are absent at $\eta k \to \infty$. For example, infrared divergences did not appear, and time proved to be an irrelevant variable in the dissipation range. Therefore, *dissipation range turbulence is not as great an obstacle for strong* coupling techniques as turbulence in the inertial range, and the renormalized perturbation expansions as well as the renormalization group technique should be tested primarily in this area.

A number of effects in the system of dissipative modes have their own interest. In particular, in section 2, the self-induced phase chaos of dissipative harmonics in flows

0305-4470/91/245763+11\$03.50

of space dimensions $d \geq 2$ is considered. This chaos will be shown to occur; it prevents marked intermittency being produced in the dissipation range. Thus, intermittency effects, which break the scaling in the inertial range, are of increasingly lesser importance as $\eta k \to \infty$. In this respect a considerable difference in fluid turbulence from one-dimensional systems should be noticed. For the latter case Frisch and Morf (1981) revealed an enhanced influence of intermittency at $\eta k \to \infty$. One may expect the scaling properties to appear in the most pure fashion in fluid turbulence at $\eta k \to \infty$.

In section 3 a non-isotropic energy cascade to wavenumbers $\eta k \gg 1$ is investigated, and a solution for the spectral tensor is obtained. In section 4 the expansion parameter of the renormalized diagram series is revealed. This parameter proved to be the energy conversion parameter, which is the nonlinear energy supply divided by the energy dissipation. In the inertial range the equivalent parameter is the square of the Reynolds number determined from the effective viscosity (see Kuz'min and Patashinskii 1972). In the dissipation range, the Reynolds number is small, but the energy conversion parameter appears to be of the order of unity, because the nonlinear inflow of energy is equal approximately to the dissipation at a given scale. Thus the dissipation range turbulence is a typical example of a system with strong interaction. A reasonable theory can be obtained by taking into account only the first few diagrams. We believe that other diagrams are of less importance in the renormalized series.

2. Phase chaos at short scales

Let us consider the spatially periodic flow of incompressible fluid at small Reynolds number. The velocity field is represented as the Fourier series

$$\boldsymbol{u}(\boldsymbol{x}, t) = \sum_{\boldsymbol{k}} \boldsymbol{u}(\boldsymbol{k}, t) \exp(\mathrm{i}\boldsymbol{k}\boldsymbol{x})$$

$$\boldsymbol{u}(\boldsymbol{k}, t) = L^{-d} \int \mathrm{d}^d x\, \boldsymbol{u}(\boldsymbol{x}, t) \exp(-\mathrm{i}\boldsymbol{k}\boldsymbol{x})$$

where L is the spatial period, which is assumed to be very large, and d is the dimension of space. From the Navier–Stokes equations, one obtains the equations for the complex amplitudes $\boldsymbol{u}(\boldsymbol{k}, t)$:

$$(\partial/\partial t + \nu k^2) u_i(\boldsymbol{k}, t) = (-\mathrm{i}/2) P_{ijl}(\boldsymbol{k}) \sum_{\boldsymbol{k}} u_j(\boldsymbol{q}, t) u_l(\boldsymbol{k}-\boldsymbol{q}, t) \tag{2.1}$$

where

$$P_{ijl}(\boldsymbol{k}) = k_j \Delta_{il}(\boldsymbol{k}) + k_l \Delta_{ij}(\boldsymbol{k}) \qquad \Delta_{ij}(\boldsymbol{k}) = \delta_{ij} - k_i k_j / k^2 \qquad \boldsymbol{k} \cdot \boldsymbol{u}(\boldsymbol{k}, t) = 0. \tag{2.2}$$

We assume that the initial Fourier amplitudes $\boldsymbol{v}(\boldsymbol{k}) = \boldsymbol{u}(\boldsymbol{k}, t_0)$ differ from zero only when $k < k_0 = l^{-1}$, where $l \ll L$ is the main scale of the flow. Because the Reynolds number R is small, the subsequent evolution of the Fourier components at $k \leq k_0$ is correctly determined by (2.1), omitting the right-hand side. The solution of the equation is

$$\boldsymbol{u}^0(\boldsymbol{k}, t) = \exp[-\nu k^2 (t - t_0)] \boldsymbol{v}(\boldsymbol{k}). \tag{2.3}$$

At wavenumbers $k > k_0$, the right-hand side of (2.1) cannot be discarded, because the nonlinear interactions serve as an energy source. In order to find nonlinear corrections to (2.3), we rewrite (2.1) as the integral equation

$$u_i(\boldsymbol{k}, t) = u_i^0(\boldsymbol{k}, t) + \int_{t_0}^{t} \mathrm{d}t' \exp[-\nu k^2(t-t')](\mathrm{i}/2)P_{ijl}(\boldsymbol{k}) \sum_{\boldsymbol{q}} u_j(\boldsymbol{q}, t')u_l(\boldsymbol{k}-\boldsymbol{q}, t'). \tag{2.4}$$

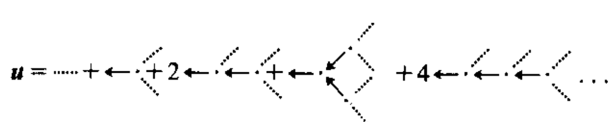
This equation can be simplified. The nonlinear interactions lead to cascade increasing of wavenumbers $k = nk_0$, and of characteristic frequencies $\omega = n\omega_0$, $\omega_0 = \nu k_0^2$ being the characteristic frequency of $\boldsymbol{u}^0$. At $k \gg k_0$, the time dependence of the velocities $u_j(\boldsymbol{q}, t')$, $u_l(\boldsymbol{k}-\boldsymbol{q}, t)$ on the right-hand side of (2.4) is slow when compared to $\exp[-\nu k^2(t-t')]$. Therefore, one can integrate over t' in (2.4), treating the velocities as time-independent. This conjecture is supported by the detailed calculations performed by Kuz'min and Patashinskii (1979). The simplified static equation for $\boldsymbol{u}(\boldsymbol{k})$ is then

$$u_i(\boldsymbol{k}) = v_i(\boldsymbol{k}) + \frac{1}{\nu k^2}\left(-\frac{\mathrm{i}}{2}\right)P_{ijl}(\boldsymbol{k}) \sum_{\boldsymbol{q}} u_j(\boldsymbol{q})u_l(\boldsymbol{k}-\boldsymbol{q}). \tag{2.5}$$

We use a graphic notation similar to that used by Wyld (1961) and Kuz'min and Patashinskii (1979). The function $(\nu k^2)^{-1}$ is represented by an arrow $\leftarrow$. The vertex operator $(-\mathrm{i}/2)P_{ijl}(\boldsymbol{k})\,\Sigma_{\boldsymbol{q}}$ is represented by a point $\cdot$, and the large-scale velocity $\boldsymbol{v}$ is represented by a line $\cdots\cdots$. Thus, (2.5) can be written symbolically as

$$\boldsymbol{u} = \cdots\cdots + \leftarrow \cdot\,{}^{\boldsymbol{u}}_{\boldsymbol{u}}. \tag{2.6}$$

Iterating (2.5) and (2.6), one obtains the velocity $\boldsymbol{u}(\boldsymbol{k})$ as a series in its large-scale component $\boldsymbol{v}(\boldsymbol{k})$. The effective small parameter of the expansion is the Reynolds number R. The graphical form of the series is given by a sum of tree diagrams:



(2.7)

At each vertex the wavevector is conserved. The sum of the wavevectors of entering lines is equal to the wavevector of the exiting arrow, so the wavevector flows without any loss from the branches $\cdots\cdots = \boldsymbol{v}$ to the trunk of a tree.

The wavevector of a diagram of nth order is equal to the sum of the wavevectors of all factors $\boldsymbol{v}(\boldsymbol{k}_i)$, where $k_i \propto k_0$:

$$\boldsymbol{k} = \sum_{i=1}^{n} \boldsymbol{k}_i \qquad |\boldsymbol{k}_i| \propto k_0. \tag{2.8}$$

The analytic expression for the nth-order diagram is of the form

$$I_n = \sum_{\boldsymbol{k}_1+\ldots+\boldsymbol{k}_n=\boldsymbol{k}} M(\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_n) \prod (\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_n) \tag{2.9}$$

where $\Pi = \Pi_{i=1}^{n} \boldsymbol{v}(\boldsymbol{k}_i)$ and M is a vertex function of nth order, which is composed of the functions $(\nu k^2)^{-1}$, and of the vertex functions $P(\boldsymbol{k})$. Indices are not shown for simplicity.

Let us treat each $\boldsymbol{k}_i$ in (2.8) as a step, and the sum (2.8) as a result of a walk in Fourier space. The sum (2.9) over all $\boldsymbol{k}_i$ is thus a sum of contributions to $\boldsymbol{u}(\boldsymbol{k})$ from different paths $\{\boldsymbol{k}_i\}$. For $k \gg k_0$, the first $n < k/k_0$ terms in the expansion (2.7) give no contribution to $\boldsymbol{u}(\boldsymbol{k})$, because the condition (2.8) can be fulfilled only if $n > k/k_0$.

The contribution of terms, the order n of which exceeds k/k_0 only slightly, is still small because the available volume, which is restricted by (2.8), is small. On the other hand, the contributions of terms of orders $n \gg k/k_0$ is small in the parameter $R \ll 1$. So there exists an optimal order $n_0 > k/k_0$ giving the maximal contribution to $u(\boldsymbol{k})$. The optimal order n_0 is produced by competition among the available volume and the power of the effective expansion parameter.

It may be concluded that the optimal n_0 corresponds to such paths that almost every step leads in the $\boldsymbol{k}$-direction, so that the longitudinal projections of $\boldsymbol{k}_i$ are positive, and are of the order of k_0. The transverse components of $\boldsymbol{k}_i$ are of the same order but have no preferred direction.

When estimating (2.9), the phases of the complex amplitudes have to be taken into account. Denoting $v_m(\boldsymbol{k}) = |v_m(\boldsymbol{k})| \exp(\mathrm{i}\phi_m(\boldsymbol{k}))$, one has $\Pi = |\Pi| \exp(\mathrm{i}\Phi)$, where $\Pi = \exp(\Sigma_{i=1}^n \log|v(\boldsymbol{k}_i)|)$, $\Phi = \Sigma_{i=1}^n \phi(\boldsymbol{k}_i)$. Let us suppose that $v(\boldsymbol{k})$ is an analytic function of $\boldsymbol{k}$. For a small variation of a path $\{\boldsymbol{k}_i + \delta\boldsymbol{k}_i\}$, the phase Φ changes additively:

$$\delta\Phi = \sum_{i=1}^{n} \delta\phi(\boldsymbol{k}_i) \qquad \delta\phi(\boldsymbol{k}_i) = \left.\frac{\partial\phi(\boldsymbol{k})}{\partial\boldsymbol{k}}\right|_{\boldsymbol{k}=\boldsymbol{k}_i} \delta\boldsymbol{k}_i. \tag{2.10}$$

Thus, at large n a small variation of a path may lead to a great variation of the total phase $\delta\Phi > \pi$. Such a behaviour of Φ implies a strong interference of contributions from different paths. Only variations inside a thin tube in $\boldsymbol{k}$-space are allowable without destroying the phase Φ.

Let us estimate the effective number of tubes with different phases. The total shift of the phase (2.10) is composed of a large number of small shifts $\delta\phi(\boldsymbol{k}_i) \propto \delta\boldsymbol{k}_i/k_0$ with arbitrary signs, so $\delta\Phi$ is estimated as in the theory of Brownian motion as $\delta\Phi \propto (\delta\boldsymbol{k}_i/k_0)\sqrt{n}$. This value is less than π if $\delta k_i < k_0/\sqrt{n}$. Thus, in (2.9) one may replace the sum over k_i, $i = 1, 2, \ldots$ by a sum over elementary cubes of volume $(\delta k_i)^d \propto (k_0/\sqrt{n})^d \propto (k_0^3/k)^{d/2}$ (note that $n \propto k/k_0$). The volume in which the factors $v(\boldsymbol{k})$ do not vanish is of the order of k_0^d, so the number of such cubes is equal to $k_0^d/(k_0^3/k)^{d/2} \propto (k/k_0)^{d/2}$.

Any two paths are considered as different only if they pass through different sets of cubes in any sequence, so any path occurs in (2.9) $n! \propto 2\pi n^{n+1/2}/\exp(n)$ times. Thus, the number of different paths $N(\boldsymbol{k})$ is of the order of

$$N(\boldsymbol{k}) \propto [(k/k_0)^{d/2}]^n/n! \propto (k/k_0)^{n(d-2)/2} \exp(n). \tag{2.11}$$

At $d \geqslant 2$, $n \propto k/k_0 \gg 1$ this number is very large.

The expression (2.9) can be written as a sum of contributions from the tubes $\{\boldsymbol{k}_i\}$:

$$I_n\{\boldsymbol{k}\} = \sum_{\{\boldsymbol{k}\}} M\{\boldsymbol{k}_i\}\Pi\{\boldsymbol{k}_i\}. \tag{2.12}$$

The phase of the contributions has been shown to be a sharp and complicated function of the path $\{\boldsymbol{k}_i\}$. Very often such a complicated function with sharp and unpredictable behaviour is identified to a random function (e.g. see Lichtenberg and Lieberman 1983). Summing up (2.12) of such random contributions may be treated as a random walk in a complex plane. Both the amplitude and phase of I_n, which are results of the random walk, are random.

From the above considerations, we suppose that only the statistical properties of the complex Fourier amplitudes $u(\boldsymbol{k})$ matter at large wavenumbers. The memory of the phases of $v(\boldsymbol{k})$ is lost when the energy transfers to wavenumbers $k \gg l^{-1}$. The same supposition seems reasonable for most of the characteristics of the amplitudes of $v(\boldsymbol{k})$.

However, some information about the orientation of the initial vortex is conserved because the random walk in $\boldsymbol{k}$-space has the preferred direction $\boldsymbol{k}$.

If most of the information about the large-scale field $\boldsymbol{v}$ is lost, we may replace it by a random field with suitable statistical characteristics. For turbulent fluctuations in the dissipation range, such a theory was studied previously by Kraichnan (1959), Kuz'min (1971), Kuz'min and Patashinskii (1979) and Dubovikov and Tatarskii (1986) with the result (1.2). We develop a similar universal theory for small-scale motions in vortices at small Reynolds number. The only complication is the loss of isotropy.

3. The exponential solution to the equation for the spectral tensor

In the previous section we examined a short-wave asymptote of the Fourier-transformed velocity at small Reynolds number. We argued that all details but isotropy of the large-scale velocity do not affect the small-scale component at $d \geqslant 2$. So the initial dynamic problem may be replaced by a more simple statistical one.

Let us consider (2.5), where $\boldsymbol{v}(\boldsymbol{k})$ is now an external random field, the source of the small-scale motion. It is assumed that the random field $\boldsymbol{v}$ is homogeneous, has normal distribution, but is not isotropic. Note that this assumption is not valid in the theory of developed turbulence at asymptotically high Reynolds numbers because of the intermittency at the Kolmogorov scale (Monin and Yaglom 1971). In particular, the local Kolmogorov scale η may fluctuate, and averaging over the fluctuations generally influences the spectrum (Kraichnan 1967, Keller and Yaglom 1970). On the other hand, our consideration is restricted by the condition $R \ll 1$. Our choice of the ensemble is related to the structure of an individual vortex packet that has the same $\langle|\boldsymbol{v}(\boldsymbol{k})|^2\rangle$.

For the Gaussian field $\boldsymbol{v}(\boldsymbol{k})$ any mean value of the type

$$\langle v_{i_1}(\boldsymbol{k}_1) \ldots v_{i_n}(\boldsymbol{k}_n)\rangle$$

can be represented by a sum of products of all possible pairwise averages (the analogue of the Wick theorem in quantum field theory). The average $\langle v_i(\boldsymbol{k}) v_j(\boldsymbol{k}')\rangle$ is represented by the Hermitian spectral tensor

$$F^0_{ij}(\boldsymbol{k}) = (L/2\pi)^d \langle v_i(\boldsymbol{k}) v_j(-\boldsymbol{k})\rangle.$$

We assume that the spectral tensor differs from zero only when $k < l^{-1}$.

Any velocity function can be expanded in a formal functional series in $\boldsymbol{v}(\boldsymbol{k})$. An example of such an expansion is the diagram series (2.7). The diagram expansion for the spectral tensor

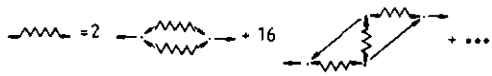
$$F_{ij}(\boldsymbol{k}) = (L/2\pi)^d \langle u_i(\boldsymbol{k}) u_j(-\boldsymbol{k})\rangle$$

is obtained after multiplying (2.7) by the similar expansion for $u_j(-\boldsymbol{k})$ and averaging over $\boldsymbol{v}(\boldsymbol{k})$. In the limit $L \to \infty$ any sums over wavevectors are replaced by integrals according to

$$(2\pi/L)^d \sum_{\boldsymbol{q}} \Rightarrow \int \mathrm{d}^d q.$$

After partial summing up of the non-renormalized diagram series, one arrives at the complete system of diagram equations for the spectral tensor F_{ij}, the response tensor and the vertex functions (Kuz'min and Patashinskii 1979).

The analysis of the equations is similar to that performed by Kuz'min and Patashinskii (1979) for the isotropic dissipation range. The response tensor and the vertices describe the external non-random perturbations which are unaffected by the weak small-scale component $\boldsymbol{u}(\boldsymbol{k})$, so these functions are assumed to coincide with the non-renormalized ones. In other words, for $k \gg l^{-1}$ the nonlinearity should be taken into account only so far as it is the only energy source. Therefore, it remains to solve the equation for the spectral tensor F_{ij}. This equation assumes the form (see equation (8) of Kuz'min and Patashinskii (1979))

 (3.1)

where the spectral tensor is represented by a wavy line . We shall seek the solution to (3.1) in the form

$$F_{ij}(\boldsymbol{k}) = \Psi_{ij}(\boldsymbol{k}) \exp[-(\eta(\boldsymbol{e})k)^{\gamma}]. \tag{3.2}$$

Here $\boldsymbol{e} = \boldsymbol{k}/k$ is the unit vector in the direction of $\boldsymbol{k}$. The function $\eta(\boldsymbol{e})$ is assumed to be determined by the condition $F_{ij} \propto F^0_{ij}$ at $k \propto l^{-1}$. We assume that $\eta(\boldsymbol{e}) = \eta(-\boldsymbol{e})$, $\gamma > 1$, Ψ_{ij} is a Hermitian tensor that varies, when $k \gg l^{-1}$, less rapidly than a power function.

Let us compute approximately the tensor F_{ij} with the aid of (3.1), on the right-hand side of which we retain only the first diagram. The equation to be solved is

$$F_{ij}(\boldsymbol{k}) = \nu^{-2}k^{-4} \int \mathrm{d}^d q (k_l F_{lm}(\boldsymbol{q}) k_m \Delta_{is}(\boldsymbol{k}) F_{sn}(\boldsymbol{k}-\boldsymbol{q}) \Delta_{nj}(\boldsymbol{k})$$
$$+ k_l F_{lm}(\boldsymbol{q}) \Delta_{mj}(\boldsymbol{k}) \Delta_{is}(\boldsymbol{k}) F_{sn}(\boldsymbol{k}-\boldsymbol{q}) k_n).$$

Substituting (3.2) into this equation, we have

$$\Psi_{ij}(\boldsymbol{k}) = \nu^{-2}k^{-4} \int \mathrm{d}^d q (k_l \Psi_{lm}(\boldsymbol{q}) k_m \Delta_{is}(\boldsymbol{k}) \Psi_{sn}(\boldsymbol{k}-\boldsymbol{q}) \Delta_{nj}(\boldsymbol{k})$$
$$+ k_l \Psi_{lm}(\boldsymbol{q}) \Delta_{mj}(\boldsymbol{k}) \Delta_{is}(\boldsymbol{k}) \Psi_{sn}(\boldsymbol{k}-\boldsymbol{q}) k_n) \exp(K) \tag{3.3}$$

where $K = (k\eta(\boldsymbol{k}))^{\gamma} - (q\eta(\boldsymbol{q}))^{\gamma} - (|\boldsymbol{k}-\boldsymbol{q}|\eta(\boldsymbol{k}-\boldsymbol{q}))^{\gamma}$. At $\eta k \gg 1$, the dominant contribution to the integral is made by the region where the index of the exponential function has its maximum value. For $\gamma > 1$, $\eta(\boldsymbol{e}) = \eta = \text{const}$, the maximum lies in the region where $\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{q} = \boldsymbol{k}/2$. It is clear that this maximum remains if non-isotropy is not too large. To define this condition more precisely, let us expand the index of the exponential function in the components of the wavevectors, which are transverse to $\boldsymbol{k}$. If we denote the non-dimensional longitudinal and transverse components of $\boldsymbol{q}$ as $\boldsymbol{s} = \boldsymbol{e}(\boldsymbol{e}\cdot\boldsymbol{q}/k)$ and $w_i = \Delta_{ij}(\boldsymbol{e}) q_j / k$, then

$$\eta^{\gamma}(\boldsymbol{q}/q) \approx \eta^{\gamma}(\boldsymbol{e}) + \frac{\partial \eta^{\gamma}}{\partial e_m}\frac{w_m}{s} + \frac{1}{2}\frac{\partial^2 \eta^{\gamma}}{\partial e_m\,\partial e_n}\frac{w_m w_n}{s^2}$$
$$\eta^{\gamma}[(\boldsymbol{k}-\boldsymbol{q})/|\boldsymbol{k}-\boldsymbol{q}|] \approx \eta^{\gamma}(\boldsymbol{e}) - \frac{\partial \eta^{\gamma}}{\partial e_m}\frac{w_m}{1-s} + \frac{1}{2}\frac{\partial^2 \eta^{\gamma}}{\partial e_m\,\partial e_n}\frac{w_m w_n}{(1-s)^2} \tag{3.4}$$
$$q^{\gamma} \approx s^{\gamma}k^{\gamma}[1 + \gamma w^2/(2s^2)] \qquad |\boldsymbol{k}-\boldsymbol{q}|^{\gamma} \approx (1-s)^{\gamma}k^{\gamma}\{1 + \gamma w^2/[2(1-s)^2]\}$$

and

$$K \approx k^\gamma \eta^\gamma(e)\left([1-s^\gamma-(1-s)^\gamma]+[(1-s)^{\gamma-1}-s^{\gamma-1}]\frac{1}{\eta^\gamma}\frac{\partial\eta^\gamma}{\partial e_m}w_m-\tfrac{1}{2}[s^{\gamma-2}+(1-s)^{\gamma-2}]\right)$$
$$\times\left(\frac{1}{\eta^\gamma}\frac{\partial^2\eta^\gamma}{\partial e_m\,\partial e_n}+\gamma\delta_{mn}\right)w_m w_n. \tag{3.5}$$

It is convenient to choose the special coordinate system in which the last axis is directed along $\boldsymbol{k}$ and the other axes are directed along the eigenvectors of the matrix

$$A_{mn}=\gamma\delta_{mn}+\frac{1}{\eta^\gamma}\frac{\partial\eta^\gamma}{\partial e_m\,\partial e_n}. \tag{3.6}$$

In this coordinate system the matrix A is as follows:

$$A=\begin{pmatrix}\lambda_1 & 0 & 0 & \dots & 0\\ & \lambda_2 & 0 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & \dots & 0 & \lambda_{d-1} & 0\\ 0 & \dots & 0 & 0 & \gamma\end{pmatrix}$$

where $\lambda_1, \lambda_2, \dots, \lambda_{d-1}, \gamma$ are the eigenvalues of the matrix.

One sees that if the eigenvalues $\lambda_1, \dots, \lambda_{d-1}$ have different signs, then the function K has a saddlepoint at $\boldsymbol{w}=0$, $s=\frac{1}{2}$. If all λ_m are positive, the function K has a maximum at this point. If any $\lambda_m=0$, terms of higher order in expansions (3.4) should be taken into account.

From (3.16) it follows that an eigenvalue λ_m may be negative if the second derivative of η^γ in the associated direction is negative and is sufficiently large in its absolute value, that is,

$$\frac{1}{\eta^\gamma}\frac{\partial^2\eta^\gamma}{\partial e_m^2}<-\gamma.$$

In these cases the dominant contribution to the integral (3.3) arises from Fourier harmonics with strongly non-collinear wavevectors. Therefore, strong interactions among the Fourier modes with different directions of their wavevectors occur. These interactions tend to diminish the strong initial non-isotropy while the energy is transferred to the high wavenumber region. One may suppose that strong non-isotropy with negative eigenvalues does not occur at $\eta k\to\infty$, though it might take place at a moderate ηk. However, arbitrary non-isotropy with positive eigenvalues is possible at any $\eta k\gg 1$. We shall not consider these cases in more detail.

For a moderate degree of non-isotropy, the eigenvalues are positive and the dominant contribution to the integral (3.3) comes from the region where $\boldsymbol{q}\approx\boldsymbol{k}/2$. Thus, the right-hand side of (3.3) is of the order of $\exp[(\eta k)^\gamma(1-2^{1-\gamma})]$, and is exponentially large compared to the left-hand side. So, for $\gamma>1$, (3.3) cannot be satisfied. For $0<\gamma<1$, the index of the exponential function has a maximum, when $\boldsymbol{q}\ll\boldsymbol{k}$ or $|\boldsymbol{k}-\boldsymbol{q}|\ll k$. This corresponds to a case in which the dominant role is played by interactions of short-wave pulsations directly with pulsations of the principal scale. However, it has been shown (Townsend 1951, Novikov 1962) that such interactions lead to a solution with $\gamma=2$, and not with $\gamma<1$. Therefore, the only γ-value that is not at variance with the equation is $\gamma=1$.

For $\gamma = 1$, (3.5) and (3.6) become

$$K = -\frac{1}{2}\frac{\eta k}{s(1-s)} A_{mn} w_m w_n \tag{3.7}$$

$$A_{mn} = \delta_{mn} + \frac{1}{\eta}\frac{\partial^2 \eta}{\partial e_m \, \partial e_n}. \tag{3.8}$$

The pre-exponential factor Ψ is obtained with the aid of the Laplace method (Erdelyi 1961). The exponent K contains the large factor $\eta k \gg 1$, and the dominant contribution to the integral on the right-hand side of (3.3) is made by the region where $\boldsymbol{q}$ and $\boldsymbol{k}$ are almost collinear. So one may expand $\Psi_{lm}(\boldsymbol{q})$, $\Psi_{sn}(\boldsymbol{k}-\boldsymbol{q})$ in (3.3) in powers of $\boldsymbol{w}$:

$$\begin{aligned}
\Psi_{lm}(\boldsymbol{q}) &= \Psi_{lm}(s\boldsymbol{k}) + k \left.\frac{\partial \Psi_{lm}(\boldsymbol{q})}{\partial q_s}\right|_{w=0} w_s + \tfrac{1}{2}k^2 \left.\frac{\partial^2 \Psi_{lm}(\boldsymbol{q})}{\partial q_s \, \partial q_r}\right|_{w=0} w_s w_r + \ldots \\
\Psi_{lm}(\boldsymbol{p}) &= \Psi_{lm}[(1-s)\boldsymbol{k}] - k \left.\frac{\partial \Psi_{lm}(\boldsymbol{p})}{\partial p_s}\right|_{w=0} w_s + \tfrac{1}{2}k^2 \left.\frac{\partial^2 \Psi_{lm}(\boldsymbol{p})}{\partial p_s \, \partial p_r}\right|_{w=0} w_s w_r - \ldots
\end{aligned} \tag{3.9}$$

where

$$\boldsymbol{q} = k(s\boldsymbol{e} + \boldsymbol{w}) \qquad \boldsymbol{p} = k[(1-s)\boldsymbol{e} - \boldsymbol{w}] \qquad \boldsymbol{e} = \boldsymbol{k}/k. \tag{3.10}$$

The solenoidality condition (2.2) implies that

$$\Psi_{ij}(\boldsymbol{q}) q_j = \Psi_{ij}(\boldsymbol{p}) p_j = 0. \tag{3.11}$$

Substituting (3.9) and (3.10) into (3.11) and equating the terms with equal powers of $\boldsymbol{w}$, one finds that

$$\begin{aligned}
& k_m \left.\frac{\partial \Psi_{im}(\boldsymbol{q})}{\partial q_s}\right|_{w=0} w_s = -\frac{1}{s}\Psi_{im}(s\boldsymbol{k}) w_m \\
& \tfrac{1}{2}k_m \left.\frac{\partial^2 \Psi_{im}(\boldsymbol{q})}{\partial q_s \, \partial q_r}\right|_{w=0} w_s w_r = -\frac{1}{s} \left.\frac{\partial \Psi_{im}(\boldsymbol{q})}{\partial q_s}\right|_{w=0} w_s w_m \\
& k_m \left.\frac{\partial \Psi_{im}(\boldsymbol{p})}{\partial p_s}\right|_{w=0} w_s = -\frac{1}{1-s}\Psi_{im}[(1-s)\boldsymbol{k}] w_m .
\end{aligned} \tag{3.12}$$

By inserting (3.7)–(3.12) into (3.3), one obtains

$$\begin{aligned}
\Psi_{ij}(\boldsymbol{k}) = \frac{1}{\nu^2 k^2} \int \mathrm{d}^d q \bigg(& \frac{1}{s^2}\Psi_{mn}(s\boldsymbol{k})\Psi_{ij}[(1-s)\boldsymbol{k}] - \frac{1}{s(1-s)}\Psi_{mj}(s\boldsymbol{k})\Psi_{in}[(1-s)\boldsymbol{k}] \bigg) \\
& \times w_m w_n \exp\left(-\frac{1}{2}\frac{\eta k}{s(1-s)} A_{rt} w_r w_t \right).
\end{aligned} \tag{3.13}$$

The integration over $\boldsymbol{w}$ gives

$$\int \mathrm{d}^{d-1} w \, w_m w_n \exp\left(-\frac{1}{2}\frac{\eta k}{s(1-s)} A_{rt} w_r w_t \right) = \delta_{mn} \frac{\pi^{(d-1)/2}[2s(1-s)]^{(d+1)/2}}{2\lambda_n \sqrt{\lambda_1 \lambda_2 \ldots \lambda_{d-1}} \, (\eta k)^{(d+1)/2}}. \tag{3.14}$$

By virtue of (3.14), (3.13) assumes the form

$$\begin{aligned}
\Psi_{ij}(\boldsymbol{k}) = & \frac{(2\pi)^{(d-1)/2} k^{d-2}}{\nu^2 \sqrt{\lambda_1 \ldots \lambda_{d-1}} \, (\eta k)^{(d+1)/2}} \int \mathrm{d}s [s(1-s)]^{(d+1)/2} \\
& \times \sum_{m=1}^{d-1} \left(\frac{1}{s^2 \lambda_m}\Psi_{mm}(s\boldsymbol{k})\Psi_{ij}[(1-s)\boldsymbol{k}] - \frac{1}{s(1-s)\lambda_m}\Psi_{mj}(s\boldsymbol{k})\Psi_{im}[(1-s)\boldsymbol{k}] \right).
\end{aligned}$$

This equation has the power solution

$$\Psi_{ij}(k)=\frac{30\nu^2(\eta k)^{(d+1)/2}\sqrt{\lambda_1\lambda_2\ldots\lambda_{d-1}}}{2^{(d-3)/2}(3d-5)k^{d-2}\pi^{(d-1)/2}}\begin{pmatrix}\lambda_1 & 0 & 0 & \ldots & 0\\ & \lambda_2 & 0 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & \ldots & 0 & \lambda_{d-1} & 0\\ 0 & \ldots & 0 & 0 & 0\end{pmatrix}.$$

Therefore, in any coordinate system the spectral tensor (3.2) can be written as

$$F_{ij}(k)=\exp[-\eta(e)k]\frac{30\nu^2(\eta k)^{(d+1)/2}\sqrt{\operatorname{Det} A}}{2^{(d-3)/2}(3d-5)k^{d-2}\pi^{(d-1)/2}}\Delta_{im}(k)A_{mn}\Delta_{nj}(k) \tag{3.15}$$

where the matrix $A_{mn}=A_{mn}(k/k)$ is defined by (3.8). If $\eta(k/k)$ does not depend on the direction of k, then $A_{mn}=\delta_{mn}$, and (3.15) reduces to the solution obtained by Kuz'min and Patashinskii (1979) and Kuz'min (1979) for the same approximation.

4. Expansion parameter and intermittency factor in the dissipation range

Let us consider diagrams of higher order. A diagram F_n, containing n integrations over d^dk, has $n+1$ wavy lines, $2n$ vertices and $2n$ functions $(\nu k^2)^{-1}$:

$$F_n\propto(P/\nu k^2)^{2n}F^{n+1}[k/(n+1)](k_\perp^{d-1}k)^n$$

where $k_\perp$ is the size of the integration domain in the transverse plane, $P\propto k_\perp$. The exponential factor restricts this size and $k_\perp$ can be estimated as

$$k_\perp\propto\sqrt{kk_0}.$$

One sees that the effective parameter of expansion (3.1) has the same order of magnitude as the first diagram in the right-hand side divided by the spectral function F. This parameter can be written as

$$\mu\propto\frac{F^2(k/2)[P^2/(\nu k^2)]k_\perp^{d-1}k}{\nu k^2F(k)}.$$

The numerator is the nonlinear supply of energy, and the denominator is the viscous dissipation. In a quasi-steady case, these factors have the same value, so $\mu\propto 1$.

Kuz'min and Patashinskii (1972, 1978) revealed a similar parameter in the inertial range. The turbulent medium was regarded as being composed of wavepackets. The Kolmogorov scaling was treated as a situation wherein the wavepackets of all scales were constructed in a similar fashion and lost an equal amount of power when overcoming the turbulent viscosity. So in the inertial range both the numerator and the denominator in μ are separate constants. The factor μ should be naturally called the energy conversion parameter. The parameter μ proved to be of the order of the Reynolds number of the wavepackets, determined from the effective viscosity. The Kolmogorov scaling corresponds to a case wherein this Reynolds number does not depend on scale and is a universal constant.

In the dissipation range the Reynolds number of the wavepackets is very small, but the energy conversion parameter μ is not. The actual role of higher-order diagrams is estimated by a direct calculation, which is possible in the dissipation range theory. Readers are referred to our analysis of the isotropic dissipation range (Kuz'min and

Patashinskii 1979), where it was shown that the approximate solution changes little when the next term in the series (3.1) is taken into account.

Similarly to Kuz'min (1979), let us consider the small-scale intermittency in the framework of the diagram technique. The small-scale component of the velocity field in the usual space is defined as

$$\boldsymbol{u}(\boldsymbol{x}, l) = \sum_{\Omega(l)} \mathrm{d}^d k\, \boldsymbol{u}(\boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k}\boldsymbol{x}).$$

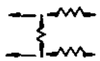
The sum is over the region $\Omega(l)$ where $k > l^{-1}$. The intermittency of the small-scale velocity $\boldsymbol{u}(\boldsymbol{k}, l)$ is determined by the flatness factor (Monin and Yaglom 1971)

$$\varkappa(l) = (\langle u_i(\boldsymbol{x}, l)^4\rangle - 3\langle u_i(\boldsymbol{x}, l)^2\rangle^2)/\langle u_i(\boldsymbol{x}, l)^2\rangle^2. \tag{4.1}$$

Let us consider the Fourier expansion of the numerator $a(l)$ in (4.1)

$$a(l) = \sum_{\boldsymbol{k}_1} \dots \sum_{\boldsymbol{k}_4} \exp[\mathrm{i}(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 + \boldsymbol{k}_4)\boldsymbol{x}](\langle u_i(\boldsymbol{k}_1) u_i(\boldsymbol{k}_2) u_i(\boldsymbol{k}_3) u_i(\boldsymbol{k}_4)\rangle$$
$$- 3\langle u_i(\boldsymbol{k}_1) u_i(\boldsymbol{k}_2)\rangle\langle u_i(\boldsymbol{k}_3) u_i(\boldsymbol{k}_4)\rangle) \tag{4.2}$$

where the summing is performed over all $\boldsymbol{k}_1 \dots \boldsymbol{k}_4$ in $\Omega(l)$. After substituting of (2.7) into (4.2), one obtains $a(l)$ as a series of all possible diagrams with four exiting lines. One of the lowest-order diagrams is



The spectral tensor decreases rapidly as its wavenumber increases. Therefore, the dominant contribution to the sum (4.2) comes from the region where the wavenumber of the internal wavy line is of the order of η^{-1}. When compared to the denominator $b(l)$ in μ

$b(l) \propto$ 

$a(l)$ contains an additional wavy line, two bare Green functions $\leftarrow \propto (\nu k^2)^{-1}$, two vertex operators $\cdot \propto k$ and one summing over the wavevectors $\boldsymbol{k} \propto \eta^{-1}$. Therefore, the flatness is of the order of

$$\varkappa(l) = a(l)/b(l) \propto (l/\eta)^2 \ll 1.$$

A similar conclusion follows from analysis of diagrams of higher orders. Thus, the diagram technique reproduces the above conclusion concerning the intermittency in the dissipation range, and the solution for the spectral tensor is self-consistent.

5. Conclusion

In this paper we analysed the energy cascade process at asymptotically high wavenumbers. We gave general arguments that the small-scale motions should have random phases, so no intermittency can build up in this region. Some amount of non-isotropy is conserved, while the energy cascades to high wavenumbers. It may be supposed that the motion in the dissipation range is composed of two components. The first one is produced by decaying coherent vortices of the Kolmogorov scale. The energy spectrum of this component may be similar to (1.1), which may be modified by intermittency effects. The second one is the universal incoherent component with the spectrum (1.2). Some similarities of the present picture to that proposed by Benzi *et al* (1986) from direct computer simulations of the two-dimensional flows should be noted. The universal analytical theory for the incoherent component is proposed and the solution for the spectral tensor is obtained.

References

Benzi R, Paladin G, Patarnello S, Santangelo P and Vulpiani A 1986 *J. Phys. A: Math. Gen.* **19** 3771
Dubovikov M M and Tatarskii V I 1986 *Zh. Eksp. Teor. Fiz.* **93** 1992
Erdelyi A 1961 *Asymptotic Expansions* (New York: Dover)
Frisch U and Morf R 1981 *Phys. Rev.* A **23** 3673
Keller B S and Yaglom A M 1970 *Mech. Jidk. Gaza* **3** 70 (in Russian)
Kraichnan R H 1959 *J. Fluid Mech.* **5** 497
—— 1967 *Phys. Fluids* **10** 2080
Kuz'min G A 1971 *Zh. Prikl. Mekh. Tekh. Fiz.* **4** 63 (Engl. transl. 1971 *J. Applied Mech. Tech. Phys.* **4** 539)
—— 1979 Dissipation range of *d*-dimensional turbulence *Preprint* 39-79, Institute of Thermophysics, Novosibirsk (in Russian)
Kuz'min G A and Patashinskii A Z 1972 *Zh. Eksp. Teor. Fiz.* **62** 1175 (Engl. transl. 1972 *Sov. Phys.–JETP* **35** 620)
—— 1978 *Zh. Prikl. Mekh. Tekh. Fiz.* **1** 62 (Engl. transl. 1978 *J. Applied Mech. Tech. Phys.* **1** 50)
—— 1979 *Zh. Eksp. Teor. Fiz.* **76** 2075 (Engl. transl. 1979 *Sov. Phys.–JETP* **49** 1050)
Lichtenberg A J and Lieberman M A 1983 *Regular and Stochastic Motion* (New York: Springer)
Monin A S and Yaglom A M 1971 *Statistical Fluid Mechanics* (Cambridge, MA: MIT Press)
Novikov E A 1962 *Sov. Phys. Dokl.* **6** 571
Townsend A A 1951 *Proc. R. Soc.* A **208** 534
Wyld H W 1961 *Ann. Phys.* **14** 449